

CONVERGENCE OF MULTIPOLE GREEN FUNCTIONS

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ABSTRACT. We continue the study of convergence of multipole pluricomplex Green functions for a bounded hyperconvex domain of \mathbb{C}^n , in the case where poles collide. We consider the case where all poles do not converge to the same point in the domain, and some of them might go to the boundary of the domain. We prove that weak convergence will imply convergence in capacity; that it implies convergence uniformly on compacta away from the poles when no poles tend to the boundary; and that the study can be reduced, in a sense, to the case where poles tend to a single point. Furthermore, we prove that the limits of Green functions can be obtained as limits of functions of the type $\max_{1 \leq i \leq 3n} \frac{1}{p} \log |f_i|$, where the f_i are holomorphic functions.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{C}^n . We say that Ω is *hyperconvex* if it admits a negative continuous plurisubharmonic exhaustion function. A *maximal* plurisubharmonic g function on a domain in \mathbb{C}^n is one that, on a small ball in the domain, lies above any plurisubharmonic function that it dominates on the boundary of that ball. Equivalently, in the case where g is locally bounded, $(dd^c g)^n = 0$, where $(dd^c)^n$ is the (complex) Monge-Ampère operator [13], [3].

Of course the Monge-Ampère operator, which potentially involves products of distributions, cannot be defined for an arbitrary locally integrable function. Bedford and Taylor [3] gave a definition for locally bounded plurisubharmonic functions. Demailly [7] extended this to plurisubharmonic functions locally bounded outside of a relatively compact set. We recall below an important class of plurisubharmonic functions introduced by Cegrell [5] on which the Monge-Ampère operator behaves nicely.

Definition 1.1. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n . We define $\mathcal{E}_0(\Omega)$ to be the class of bounded plurisubharmonic functions u on Ω such that $\lim_{z \rightarrow \partial\Omega} u(z) = 0$ and $\int_{\Omega} (dd^c u)^n < \infty$. More generally, $\mathcal{F}(\Omega)$ is the set of plurisubharmonic functions u on Ω such that there exists a sequence $u_j \in \mathcal{E}_0(\Omega)$ satisfying $u_j \downarrow u$ and $\sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n < \infty$.*

The Monge-Ampère operator is well defined on $\mathcal{F}(\Omega)$ and enjoys basic properties like continuity under monotone sequences, comparison principle, etc.

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Green functions on a hyperconvex domain Ω are fundamental solutions of the (complex) Monge-Ampère operator $(dd^c)^n$, i.e. functions G_a such that $(dd^c G_a)^n = \delta_a$, the Dirac mass at $a \in \Omega$, with zero boundary values; when Ω is hyperconvex, they are continuous up to the boundary [7], [12]. Since the operator is non-linear when $n \geq 2$, if we want $(dd^c G)^n$ to be a sum of Dirac masses, we cannot add up Green functions. A function G as above is called multipole Green function, and its study, initiated by Lelong [12], is more delicate.

Let $S := \{a_1, \dots, a_N\}$ be a finite subset of Ω . The Green function of Ω with the pole set S is defined as follows [12]:

$$G_{\Omega, S}(z) := \sup\{u(z) : u \in PSH^-(\Omega), u(z) \leq \log |z - a| + O(1), \forall a \in S\},$$

where $PSH^-(\Omega)$ stands for the set of all nonpositive plurisubharmonic functions on Ω . Multipole Green functions belong to Cegrell's class $\mathcal{F}(\Omega)$.

A multipole Green function depends continuously on its poles provided they do not collide. The following result is due to Blocki. In the case of a single pole, this had been proved by Demailly [7].

Proposition 1.2. *If Ω is hyperconvex then the map $(z, p_1, \dots, p_k) \mapsto G_{\Omega, (p_1, \dots, p_k)}(z)$ is continuous as a function defined on the set $\{\overline{\Omega} \times \Omega^k : z \neq p_j \neq p_k\}$.*

We are interested in what may happen to limits of sequences of multipole Green functions. When poles collide, new singularities will arise, generalizing the concept of multiple poles, see [14].

We establish some terminology about convergence of finite sets.

Definition 1.3. *Let Ω be a bounded domain in \mathbb{C}^n and $N \geq 1$. We say that a sequence $\{S_k\}_{k \geq 1} = \{(a_{1,k}, \dots, a_{N,k})\}_{k \geq 1} \subset \Omega^N$ is convergent if $a_{i,k} \neq a_{j,k}$ for every $1 \leq i < j \leq N$ and if S_k converges to an element $S \in \overline{\Omega}^N$; $\{S_k\}$ is called interior convergent if $S \in \Omega^N$; it is said to be boundary convergent if $S \in (\partial\Omega)^N$.*

Remarks.

- (a) Note that coordinates of the limit point of the sequence $\{S_k\}$ are not necessarily distinct.
- (b) We denote by π_N the projection $\Omega^N \rightarrow 2^\Omega$ defined by $(a_1, \dots, a_N) \mapsto \{a_1, \dots, a_N\} \subset \Omega$. We will drop the subscript N in case there is no confusion.
- (c) When the coordinates of the N -tuple S are distinct points in Ω , by a slight abuse of notation, we will write $G_{\pi(S)} = G_S$.
- (d) Renumbering the points as needed, every convergent sequence S_k in Ω^N can be partitioned as $S_k = (S'_k, S''_k)$, where S'_k (resp. S''_k) is a interior convergent (resp. boundary convergent) in $\Omega^{N'}$ (resp. $\Omega^{N''}$), with $N' + N'' = N$.

Several notions of convergence of functions will occur in this paper. We need a definition, which originates in [19].

Definition 1.4. *For a Borel subset E of Ω , the relative capacity $C(E, \Omega)$ is defined [3] as*

$$C(E, \Omega) := \sup\left\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 < u < 0\right\}.$$

Given a sequence of functions $\{u_k\}$, we say $u_k \rightarrow 0$ in capacity on Ω if for every $\varepsilon > 0$ and every Borel set $F \Subset \Omega$ we have $C(\{z \in F : |u_k(z)| > \varepsilon\}, \Omega) \rightarrow 0$ as $k \rightarrow \infty$.

It is clear that uniform convergence on compacta of Ω implies convergence in capacity. In fact, uniform convergence on compacta of $\Omega \setminus E$, where E is a compact set with zero capacity (e.g. a finite set) is enough to imply convergence in capacity. On the other hand, convergence in capacity implies convergence in measure (in the sense of Lebesgue measure). If a sequence converges in measure and in the L_{loc}^1 topology, both limits must coincide.

By weak compactness in the L_{loc}^1 topology, we always have limit points for a sequence of multipole Green functions G_{S_k} . Previous work [17] gave sufficient algebraic conditions for such convergence.

In the same paper, [17, Theorem 3.1] proved that for sequences of Green functions with all poles tending to one point in Ω , convergence in the L_{loc}^1 topology (the weakest possible in a sense) implies the much stronger uniform convergence on compacta. The following generalizes [17, Theorem 3.1].

Theorem 1.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\{S_k\}_{k \geq 1}$ be a sequence that converges to $S = (s_1, \dots, s_N) \in \overline{\Omega}^N$. Suppose that $G_k := G_{\Omega, S_k}$ converges in L_{loc}^1 to a plurisubharmonic function g on Ω . Then the following assertions hold:*

- (a) G_k converges in capacity to g on Ω .
- (b) g is continuous and maximal plurisubharmonic on $\Omega \setminus \pi(S)$, and $\lim_{z \rightarrow \partial\Omega} g(z) = 0$.
- (c) $(dd^c g)^n = \sum_{a \in \pi(S) \cap \Omega} \nu_a \delta_a$, where $\nu_a := \#\{j \in \{1, \dots, N\} : s_j = a\}$.

Furthermore, if $\{S_k\}_{k \geq 1}$ is an interior convergent sequence, the convergence is also uniform on compacta of $\overline{\Omega} \setminus \pi(S)$.

The special case of boundary convergent sequences is simpler, in that no assumption is needed on the convergence of the Green functions. As an immediate consequence of [11, Theorem 3.5], we have:

Proposition 1.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Let $\{u_k\} \subset \mathcal{F}(\Omega)$ be a sequence satisfying the following properties:*

- (a) $\sup_{k \geq 1} \int_{\Omega} (dd^c u_k)^n < \infty$;
- (b) $\int_E (dd^c u_k)^n \rightarrow 0$ for every Borel set $E \Subset \Omega$ as $k \rightarrow \infty$.

Then $u_k \rightarrow 0$ in capacity on Ω .

Applying this to $u_k = G_{S_k}$, where $\int_{\Omega} (dd^c u_k)^n = N$ for all k , we have:

Corollary 1.7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\{S_k\}_{k \geq 1} \subset \Omega^N$ a boundary convergent sequence. Then G_{S_k} converges to 0 in capacity.*

In connection with Corollary 1.7, the following question arises naturally: if $S_k = (s_{1,k}, \dots, s_{N,k}) \subset \Omega^N$ tends to $S = (s_1, \dots, s_N) \in (\partial\Omega)^N$, when does G_{Ω, S_k} converge uniformly on compact sets of Ω ? Then the conclusions of Theorem 1.5 and Corollary 1.8 could be strengthened to convergence on compacta of $\Omega \setminus S$ instead of convergence in capacity. By the rough estimate

$$G_{\Omega, S_k} \geq G_{\Omega, s_{1,k}} + \dots + G_{\Omega, s_{N,k}},$$

it suffices to consider the case $N = 1$.

There is a folklore conjecture that the answer is always positive when Ω is bounded hyperconvex. Nevertheless, it is only proved under the assumptions that Ω admits a negative plurisubharmonic exhaustion function which is either Hölder continuous [9] or satisfies certain mild conditions about the growth near the boundary [10].

A closely related problem is about uniform convergence of G_{Ω, S_k} on compacta of $\overline{\Omega} \setminus S$. This question has been studied by Coman (for $N = 1$, which is enough). He shows [6, Theorem 5] that if Ω admits a plurisubharmonic peaking function at the cluster point $s \in \partial\Omega$ which is also Hölder continuous near s , then G_{Ω, S_k} does converge uniformly on compact sets of $\overline{\Omega} \setminus \{s\}$. In particular, this is true at every point $s \in \partial\Omega$ if Ω is sufficiently regular (e.g. a real-analytic pseudoconvex domain).

In the general case, we can apply Theorem 1.5 to give a characterization for convergence in capacity of the sequence G_{Ω, S_k} .

Corollary 1.8. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and $\{S_k\}_{k \geq 1}$ be a sequence that converges to $S = (s_1, \dots, s_N) \in \overline{\Omega}^N$. Let*

$$G := (\limsup_{k \rightarrow \infty} G_{\Omega, S_k})^*.$$

Assume that

$$(dd^c G)^n(\pi(S) \cap \Omega) \geq \#\{j : s_j \in \Omega\}.$$

Then G_{Ω, S_k} converges in capacity to G on Ω as $k \rightarrow \infty$.

If S_k is interior convergent, the convergence is locally uniform on $\overline{\Omega} \setminus \pi(S)$.

Finally, for interior convergent sequences, we can reduce the study of convergence of multipole Green functions to what happens in the case of a single limit point for the poles by using systematically the next result, which shows how we can break up the limit set $\pi(S)$ into smaller pieces.

Proposition 1.9. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\{S_k\}$ be an interior convergent sequence in Ω^N . Assume that $S_k = (S'_k, S''_k)$ where $\{S'_k\}_{k \geq 1}$ and $\{S''_k\}_{k \geq 1}$ are interior convergent sequences of $\Omega^{N'}$ and $\Omega^{N''}$ respectively ($N' + N'' = N$). Suppose that $S'_k \rightarrow a' \in \Omega^{N'}$, $S''_k \rightarrow a'' \in \Omega^{N''}$, and that $\pi_{N'}(a') \cap \pi_{N''}(a'') = \emptyset$. Then the following statements are equivalent:*

- (a) G_{Ω, S_k} converges in L^1_{loc} on $\Omega \setminus \pi_N(S)$.
- (b) G_{Ω, S_k} converges locally uniformly on $\Omega \setminus \pi_N(S)$.
- (c) The two sequences G_{Ω, S'_k} and G_{Ω, S''_k} converges in L^1_{loc} on $\Omega \setminus \pi_{N'}(S')$ and $\Omega \setminus \pi_{N''}(S'')$ respectively.
- (d) The two sequences G_{Ω, S'_k} and G_{Ω, S''_k} converges locally uniformly on $\Omega \setminus \pi_{N'}(S')$ and $\Omega \setminus \pi_{N''}(S'')$ respectively.

Furthermore, when convergence occurs, if we write $g := \lim_k G_{\Omega, S_k}$, $g' := \lim_k G_{\Omega, S'_k}$, $g'' := \lim_k G_{\Omega, S''_k}$, then we have the following relation:

$$g = \sup \{u \in PSH^-(\Omega) : u \leq g' + O(1), u \leq g'' + O(1)\}.$$

For a finite set $S \subset \Omega$, we will denote by $\mathcal{I}_{\Omega,S}$ the ideal of holomorphic functions on Ω which vanish on the set S . The p -th power of that ideal, $\mathcal{I}_{\Omega,S}^p$, is the ideal of holomorphic functions in Ω which vanish to order at least p on the set S .

It is natural to ask how close a pluricomplex Green function with pole set S is to being a maximum of functions of the form $\frac{1}{p} \log |f|$, where $f \in \mathcal{I}_{\Omega,S}^p$. A version of this in the framework of the convergence question is given below; it is inspired by [15, Theorem 1.1]. First we need to define a slightly more restrictive class of domains.

Definition 1.10. *A bounded domain Ω in \mathbb{C}^n is said to be strictly hyperconvex if there exist a bounded open neighbourhood U of $\overline{\Omega}$ and a real valued continuous plurisubharmonic function ρ on U such that $\Omega = \{z \in U : \rho(z) < 0\}$.*

This type of domains, under slightly stronger condition, was introduced earlier in [15]. Note that there exist smoothly bounded pseudoconvex domains which do not have a Stein neighbourhood basis (e.g., worm-domains), in particular, such domains are hyperconvex but not strictly hyperconvex.

Theorem 1.11. *Let Ω be a bounded strictly hyperconvex domain in \mathbb{C}^n and $\{S_k\}$ be an interior convergent sequence Ω^N that converges to $S \in \Omega^N$. Suppose that G_{Ω,S_k} tends to $g \in PSH^-(\Omega)$ in $L_{loc}^1(\Omega)$. Then for every $\varepsilon > 0$, there exist a hyperconvex domain Ω_ε containing $\overline{\Omega}$ such that for every compact $K \subset \Omega \setminus S$, we can find $p \geq 1$ and a collection of holomorphic functions $\{(f_{1,k}, \dots, f_{3n,k})\}_{k \geq 1}$ where $f_{j,k} \in \mathcal{I}_{\Omega_\varepsilon, S_k}^p$ satisfying the following conditions:*

- (i) $\|f_{j,k}\|_{\overline{\Omega}} < 1$;
- (ii) *For every k large enough, the following estimate holds on K*

$$g - \varepsilon < \frac{1}{p} \max\{\log |f_{1,k}|, \dots, \log |f_{3n,k}|\} < g + \varepsilon.$$

- (iii) *The common zero set in Ω_ε of $f_{1,k}, \dots, f_{3n,k}$ coincides exactly with S .*

In the converse direction we have the following partial result, which says that if a sequence of functions of the correct type converges, then it must converge to the limit of the corresponding Green functions.

Proposition 1.12. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and $\{S_k\}_{k \geq 1}$ be a sequence in Ω^N that converges to $S \in \Omega^N$. Assume that there exist a sequence $\{p_k\}_{k \geq 1}$ of positive integers, a collection $\{(f_{1,k}, \dots, f_{n_k,k})\}_{k \geq 1}$ of holomorphic functions on Ω satisfying the following properties.*

- (i) $\|f_{j,k}\|_{\overline{\Omega}} < 1$ for every j, k .
- (ii) $f_{j,k} \in \mathcal{I}_{S_k}^{p_k}$ for every k and $1 \leq j \leq n_k$.
- (iii) *The sequence $u_k := \frac{1}{p_k} \max\{\log |f_{1,k}|, \dots, \log |f_{n_k,k}|\}$ converges in $L_{loc}^1(\Omega)$ to $u \in \mathcal{F}(\Omega)$ as k tends to ∞ .*
- (iv) $\int_{\Omega} (dd^c u)^n \leq N$.

Then the sequence G_{Ω, S_k} converges to u locally uniformly on $\overline{\Omega} \setminus \pi(S)$.

2. PROOFS OF THEOREM 1.5, COROLLARY 1.8, AND PROPOSITION 1.9

Our first objective is to show how, for the very special case of Green functions with interior convergent pole sets, L_{loc}^1 convergence implies uniform convergence on compacta. In order to do so, we will need a property of uniform continuity that will first require a lemma about variation of domains.

Lemma 2.1. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n and E be a finite subset of Ω . Then for every $\delta > 0$, there exists $r_0 > 0$ and a relatively compact hyperconvex subdomain Ω' of Ω such that for every pole set $E' \subset \cup_{a \in E} \mathbb{B}(a, r_0)$ we have*

$$G_{\Omega', E'}(z) \leq G_{\Omega, E'}(z) + \delta, \quad \forall z \in \Omega'.$$

Proof. Choose a compact hyperconvex subdomain Ω' of Ω such that for every $\alpha \in E$ we have

$$G_{\Omega, \alpha}(z) > -\frac{\delta}{2N}, \quad \forall z \in \partial\Omega',$$

where $N := \#E$. By continuity with respect to the pole of the Green functions (cf. Proposition 1.2) we can find $r_0 > 0$ so small that for every $a \in \cup_{\alpha \in E} \mathbb{B}(\alpha, r_0)$

$$G_{\Omega, a}(z) > -\frac{\delta}{N}, \quad \forall z \in \partial\Omega'.$$

Thus for $E' \subset \cup_{\alpha \in E} \mathbb{B}(\alpha, r_0)$ with $\#E' = N$ we have

$$G_{\Omega, E'}(z) \geq \sum_{\alpha \in E'} G_{\Omega, \alpha}(z) > -\delta, \quad \forall z \in \partial\Omega'.$$

It follows that $G_{\Omega', E'} \leq G_{\Omega, E'} + \delta$ on $\partial\Omega'$. Define $\hat{G} = \max\{G_{\Omega', E'} - \delta, G_{\Omega, E'}\}$ on Ω' and $\hat{G} = G_{\Omega, E'}$ on $\Omega \setminus \Omega'$. It follows from the definition of Green function that $\hat{G} \leq G_{\Omega, E'}$ on Ω . By restricting this inequality to Ω' , we obtain the desired estimate. \square

Lemma 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\{S_k\}_{k \geq 1} \subset \Omega^N$ be a sequence that converges to $S \in \Omega^N$. Then for every fixed point $z_0 \in \Omega \setminus \pi(S)$ and $\delta > 0$, there exist $r_0 > 0, k_0 \geq 1$ such that for every $z', z'' \in B(z_0, r_0)$ and $k \geq k_0$ we have*

$$|G_{\Omega, S_k}(z') - G_{\Omega, S_k}(z'')| < \delta.$$

Proof. Let $z \cdot \bar{w} := \sum_k z_k \bar{w}_k$ stand for the Hermitian inner product in \mathbb{C}^n . We pick $u \in \mathbb{S}^{2n-1} := \{v \in \mathbb{C}^n : \|v\| = 1\}$ such that for any $a \in \pi(S)$, the orthogonal projection of a to $z_0 + \mathbb{C}u$, $\pi_{z, u}(a) := z_0 + ((a - z_0) \cdot \bar{u})u$ is different from z_0 . This is possible since $\bigcup_{a \in \pi(S)} \{u \in \mathbb{S}^{2n-1} : (a - z_0) \cdot \bar{u} = 0\}$ is a subvariety of real codimension 2 of \mathbb{S}^{2n-1} .

Let $r_1 := \min_{a \in \pi(S)} |(a - z_0) \cdot \bar{u}|$. For $k \geq k_1$, $\min_{a \in \pi(S_k)} |(a - z_0) \cdot \bar{u}| \geq \frac{2}{3}r_1$. Let $r_0 := \frac{1}{3}r_1$. If we take $z' \in B(z_0, r_0)$, then

$$|(z' - a) \cdot \bar{u}| = |(z' - z_0) \cdot \bar{u} - (a - z_0) \cdot \bar{u}| \geq r_0.$$

We define for $k \geq 1$

$$P_k(z) := \prod_{a \in S_k} ((z - a) \cdot \bar{u}),$$

and for $k \geq k_1$,

$$\Phi_k(z) := z + \frac{P_k(z)}{P_k(z')}(z'' - z').$$

Note that $\left| \frac{P_k(z)}{P_k(z')} \right| \leq C := (2\text{diam}(\Omega)/r_0)^{\#(\pi(S))}$, and so $|\Phi_k(z) - z| \leq 2Cr_0$, for any $z \in \Omega$.

Using Lemma 2.1, we can find a relatively compact hyperconvex subdomain Ω' of Ω such that $z_0 \in \Omega'$, and for every k sufficiently large,

$$(1) \quad G_{\Omega', S_k}(z) < G_{\Omega, S_k}(z) + \delta, \quad \forall z \in \Omega'.$$

Now, reducing the value of r_0 , if needed, we may assume that $B(z_0, r_0) \subset \Omega'$ and for every $z', z'' \in \mathbb{B}(z_0, r_0)$ we have $\Phi_k(\Omega') \subset \Omega$. Then, by the decreasing property under holomorphic maps of the Green functions (noting that Φ_k fixes S_k) we get

$$(2) \quad G_{\Omega, S_k}(z'') \leq G_{\Phi_k(\Omega'), S_k}(z'') \leq G_{\Omega', S_k}(z').$$

Combining (2) and (3) we arrive at

$$G_{\Omega, S_k}(z'') < G_{\Omega, S_k}(z') + \delta.$$

By exchanging the roles of z', z'' we obtain the desired estimate. \square

Lemma 2.3. *Let $\Omega, \{S_k\}_{k \geq 1}$ and S be as in Lemma 2.2. Assume that G_{Ω, S_k} tends to a plurisubharmonic function g in $L^1_{loc}(\Omega)$. Then the convergence is uniform on every compact subset of $\overline{\Omega} \setminus \pi(S)$.*

Proof. First we claim that the uniform convergence holds on compact subsets of $\Omega \setminus \pi(S)$. Assume this is false, then there exist a compact subset K of $\Omega \setminus S$, a constant $\delta > 0$, two sequences $n_k, m_k \uparrow \infty$ and a sequence $\{z_k\} \subset K$ such that

$$(3) \quad |G_{\Omega, S_{n_k}}(z_k) - G_{\Omega, S_{m_k}}(z_k)| > \delta.$$

By compactness of K , we may assume that $z_k \rightarrow z^* \in K$. By passing to subsequences we can suppose further that both sequences $G_{\Omega, S_{n_k}}$ and $G_{\Omega, S_{m_k}}$ converge *pointwise* to g on a dense subset of Ω . According to Lemma 2.2, there exists $r_0 > 0$ such that for every $z', z'' \in \mathbb{B}(z^*, r_0)$ and k large enough we have

$$(4) \quad |G_{\Omega, S_{n_k}}(z') - G_{\Omega, S_{n_k}}(z'')| < \delta/3, |G_{\Omega, S_{m_k}}(z') - G_{\Omega, S_{m_k}}(z'')| < \delta/3.$$

Choose a point $w \in B(z^*, r_0)$ such that for k large enough we have

$$(5) \quad |G_{\Omega, S_{n_k}}(w) - G_{\Omega, S_{m_k}}(w)| < \delta/3.$$

By applying (5) with $z' = z_k, z'' = w$, together with (4) and (6) we reach a contradiction. This proves our claim.

To get the convergence up to the boundary, we need to use a property of sequences of Green functions that stems from the form of their singularities and the fact that they are maximal plurisubharmonic outside their poles. We follow [14, Lemma 4.5].

Lemma 2.4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\{S_k\}_{k \geq 1}$ be sequence that converges to $S \in \Omega^N$. Denote $G_k := G_{\Omega, S_k}$. Assume that there exist constants*

$\delta_0 > 0, C > 0$ with the following property: For every $\delta \in (0, \delta_0]$ we can find $k(\delta) > 0$ such that for every $k_1 > k_2 > k(\delta)$ and $z \in \cup_{a \in \pi(S)} \partial B(a, \delta)$ we have

$$|G_{k_1}(z) - G_{k_2}(z)| \leq C.$$

Then the sequence $\{G_k\}$ converges uniformly on compact sets of $\overline{\Omega} \setminus \pi(S)$.

Proof. We can assume that $\overline{B}(a, \delta_0) \cap \overline{B}(a', \delta_0) = \emptyset$ for $a \neq a' \in \pi(S)$. Given a compactum $K \subset \overline{\Omega} \setminus \pi(S)$, we have uniform bounds on our Green functions given by $0 \geq G_k \geq \sum_{s \in S_k} G_{\Omega, s}$, so that to prove a uniform Cauchy condition, it will be enough to prove that for any $\eta \in (0, 1)$, $z \in K$, there exists $k(\eta)$ such that for $k_1, k_2 \geq k(\eta)$,

$$(6) \quad (1 + \eta)G_{k_1}(z) \leq G_{k_2}(z) \leq (1 - \eta)G_{k_2}(z).$$

Near any of its poles $a \in E$, a Green function verifies $G_{\Omega, E}(z) \leq \log \|z - a\| - \log r$, if $B(a, r) \subset \Omega$. Fix $\eta \in (0, 1)$, using the hypothesis and this upper bound, we can find $\delta(\eta)$ so small such that for $\delta \in (0, \delta(\eta))$ there exists $k(\delta)$ such that if $k_1 > k_2 > k(\delta)$, the inequalities (6) hold for $z \in \cup_{a \in \pi(S)} \partial B(a, \delta)$. They continue to hold on $\Omega \setminus \cup_{a \in \pi(S)} \overline{B}(a, \delta)$ since G_{k_1} and G_{k_2} are both maximal continuous plurisubharmonic there, and tend to 0 near $\partial\Omega$. \square

To finish the proof of Lemma 2.3, simply observe that uniform convergence on the union of spheres centered on $\pi(S)$ yields the estimate needed to apply Lemma 2.4. \square

Proof of Theorem 1.5. Following Remark (d) after Definition 1.3, we write $S_k = (S'_k, S''_k)$, where S'_k and S''_k are interior and boundary convergent sequences in $\Omega^{N'}$ and $\Omega^{N''}$, respectively. Then, for any $z \in \Omega$ and $k \geq 1$,

$$(7) \quad G_{\Omega, S''_k} + G_{\Omega, S'_k} \leq G_{\Omega, S_k} \leq G_{\Omega, S'_k}.$$

By Corollary 1.7, G_{Ω, S''_k} goes to 0 in capacity, in particular G_{Ω, S''_k} converges to 0 in L^1_{loc} . Since (7) can be rewritten

$$G_{\Omega, S_k} \leq G_{\Omega, S'_k} \leq G_{\Omega, S_k} - G_{\Omega, S''_k},$$

the assumption of the theorem implies that G_{Ω, S'_k} converges to g in L^1_{loc} . Since S'_k is interior convergent, by Lemma 2.3, the convergence is actually uniform on compact subsets of $\overline{\Omega} \setminus \pi(S')$, where $S' := \lim_{k \rightarrow \infty} S'_k$. In particular, we have $\lim_{z \rightarrow \partial\Omega} g(z) = 0$, and also the last statement of the theorem (which is the case where $S_k = S'_k$).

Since uniform convergence of G_{Ω, S'_k} on compacta of $\Omega \setminus \pi(S)$ implies its convergence in capacity, (7) implies conclusion (a) of the theorem.

For (c), it suffices to repeat the reasoning at the end of the proof of [17, Theorem 1.1]. We omit the details. \square

In the proof of Corollary 1.8, we will need the following properties of functions in the class $\mathcal{F}(\Omega)$.

Lemma 2.5. *Let $u, v \in \mathcal{F}(\Omega)$ with $u \leq v$. Then the following assertions hold.*

- (a) $\int_S (dd^c v)^n \leq \int_S (dd^c u)^n$ for every pluripolar subset S of Ω .
- (b) $\int_\Omega (v - u)^n (dd^c w)^n \leq \int_\Omega -w[(dd^c u)^n - (dd^c v)^n]$, for every $w \in PSH(\Omega)$, $-1 \leq w < 0$.

Proof. For (a), see [2, Lemma 2.1], and for (b), see [11, Proposition 3.4]. \square

Proof of Corollary 1.8. We let $g \in PSH^-(\Omega)$ be an arbitrary limit point of $G_k := G_{\Omega, S_k}$ in $L^1_{loc}(\Omega)$. Then, by Theorem 1.5, $(dd^c g)^n$ is supported on $\pi(S) \cap \Omega$ and

$$\int_{\Omega} (dd^c g)^N = \sum_{a \in \pi(S) \cap \Omega} \nu_a = \#\{j \in \{1, \dots, N\} : s_j \in \Omega\}.$$

In particular $g \in \mathcal{F}(\Omega)$. Since $g \leq G < 0$ on Ω we infer that $G \in \mathcal{F}(\Omega)$. We claim that $G = g$ on $\Omega \setminus \pi(S \cap \Omega^N)$. Assume that there exists $z_0 \in \Omega \setminus \pi(S \cap \Omega^N)$ such that $G(z_0) > g(z_0)$. Then we can find a $\delta > 0$ small enough such that:

- (i) $B(a, \delta) \cap B(a', \delta) = \emptyset$ for $a \neq a' \in \pi(S \cap \Omega^N)$;
- (ii) $B(z_0, \delta) \cap X_{\delta} := \cup_{a \in \pi(S)} \overline{B}(a, \delta)$;
- (iii) X_{δ} is polynomially convex.

To see (iii), observe that since $\pi(S)$ is a finite set in \mathbb{C}^n , there exists a complex line l passing through 0 such that the orthogonal projections of the points of $\pi(S)$ on l are all distinct (take l so that it is not orthogonal to any of the lines defined by pairs of distinct points in $\pi(S)$). Then for δ small enough, the projection of $\pi(X_{\delta})$ on H consists of a finite number of pairwise disjoint closed discs in l . Thus, by Kallin's lemma [18, Theorem 1.6.19], the set X_{δ} is polynomially convex.

This polynomial convexity allows us to choose a plurisubharmonic function w on \mathbb{C}^n such that $-1 \leq w < 0$ on Ω , w is strictly plurisubharmonic on $B(z_0, \delta)$ and $w = -1$ on X_{δ} . Using Lemma 2.5 and the fact that $(dd^c g)^n$ is supported on $\pi(S)$ we obtain

$$\int_{\Omega} (G - g)^n (dd^c w)^n \leq \int_{\Omega} w [(dd^c G)^n - (dd^c g)^n] \leq -(dd^c G)^n(\pi(S)) + (dd^c g)^n(\pi(S)) \leq 0.$$

This implies that

$$\int_{B(z_0, \delta)} (G - g)(dd^c w)^n = 0.$$

So $G = g$ a.e. on $B(z_0, \delta)$. Since both functions are plurisubharmonic they must coincide on $B(z_0, \delta)$. We get a contradiction. The proof is complete.

If S_k is interior convergent then, since G_{Ω, S_k} tends to G in L^1_{loc} , Theorem 1.5 tells us that the convergence is locally uniform on $\overline{\Omega} \setminus \pi(S)$. \square

Proof of Proposition 1.9.

By Theorem 1.5, it remains to prove the equivalence of (b) and (d).

(b) \Rightarrow (d) It suffices to show that G_{Ω, S'_k} converges locally uniformly on $\Omega \setminus \pi_{N'}(S')$. As in the proof of Theorem 1.5, we deduce from (7) that for every $z \in \Omega$ and for every k ,

$$G_{\Omega, S_k}(z) \leq G_{\Omega, S'_k}(z) \leq G_{\Omega, S_k}(z) - G_{\Omega, S''_k}(z).$$

Since the G_{Ω, S''_k} are uniformly bounded from below on a small neighbourhood of S'_k (estimating them from below by sums of one-pole Green functions), by Lemma 2.4 we get the desired conclusion.

(d) \Rightarrow (b) We use the same reasoning as above, together with the following modified form of (7):

$$(8) \quad G_{\Omega, S'_k} + G_{\Omega, S''_k} \leq G_{\Omega, S_k} \leq \min\{G_{\Omega, S'_k}, G_{\Omega, S''_k}\}.$$

To prove the last statement of the theorem, let

$$\tilde{g} := \sup \{u \in PSH^-(\Omega) : u \leq g' + O(1), u \leq g'' + O(1)\}.$$

Notice that (8) implies immediately that $g \leq \min(g', g'')$, so that g is a candidate for the upper bound which defines \tilde{g} , therefore $g \leq \tilde{g}$.

To see the reverse inequality, since $g \geq g' + g''$, in a neighborhood of S' , $g \geq g' - O(1)$, and in a neighborhood of S'' , $g \geq g'' - O(1)$. So $\tilde{g} \leq \sup \{u \in PSH^-(\Omega) : u \leq g + O(1)\}$. But an easy argument using the maximality of g , or the application of [16, Lemma 4.1], shows that the latter function is equal to g . \square

Remark. For example, we can apply this result to describe accurately the situations of convergence or non-convergence when S_k consist of 4 poles and each subset S'_k, S''_k consists of 2 poles. Indeed, suppose that $S_k = (s_{1,k}, \dots, s_{4,k}) \in \Omega^4$, that $s_{1,k}, s_{2,k} \rightarrow a' \in \Omega$, and $s_{3,k}, s_{4,k} \rightarrow a'' \in \Omega \setminus \{a'\}$. Then the Green function $G_{S_k, \Omega}$ will converge to a limit if and only if the directions $[s_{1,k} - s_{2,k}]$ and $[s_{3,k} - s_{4,k}]$ both converge in $\mathbb{P}^{n-1}\mathbb{C}$ [14, Section 6.1].

3. PROOF OF THEOREM 1.11 AND PROPOSITION 1.12.

We need a result analogous to Lemma 2.1.

Lemma 3.1. *Let Ω be a strictly hyperconvex domain in \mathbb{C}^n , K a compact subset of Ω and $N \geq 1$. Then for every $\delta > 0$, there exists a hyperconvex domain Ω_ε which contains $\overline{\Omega}$ such that for every pole set $S \subset K^N$ we have*

$$G_{\Omega, S}(z) - \delta < G_{\Omega_\varepsilon, S}(z) \leq G_{\Omega, S}(z) \quad \forall z \in \Omega \setminus S.$$

Proof. This lemma in the case where $N = 1$ and K is a single point was essentially proved by Nivoche [15, Proposition 2.3]. We will use an idea from Theorem 4.3 in [7]. For $\varepsilon > 0$, we let Ω_ε be the connected component of $\{z \in U : \rho < \varepsilon\}$ that contains Ω . By choosing ε small enough we may assure that Ω_ε is hyperconvex and relatively compact in U . We claim that if $\varepsilon > 0$ is sufficiently small then

$$G_{\Omega_\varepsilon, a} > -\delta/N, \quad \forall z \in \partial\Omega, \forall a \in K.$$

Choose $r_0 > 0$ so that $V := \cup_{a \in K} B(a, r_0)$ is relatively compact in Ω . Let d be the diameter of U . Choose a constant $C > 0$ big enough such that

$$C \sup_V \rho < \delta/N + \log(r_0/d).$$

Choose $\varepsilon > 0$ such that

$$C \sup_V \rho - \log(r_0/d) < C\varepsilon < \delta/N.$$

Consider the function

$$(9) \quad \hat{\rho}(z) := \begin{cases} \max\{C(\rho(z) - \varepsilon), \log \frac{|z-a|}{d}\} & z \in \Omega_\varepsilon \setminus B(a, r_0) \\ \log \frac{|z-a|}{d} & z \in B(a, r_0). \end{cases}$$

By the choices of C, ε we can check that $\hat{\rho} \in PSH^-(\Omega_\varepsilon)$ and $\hat{\rho}$ has logarithmic singularity at a . It follows that for every $z \in \partial\Omega$ we have

$$G_{\Omega_\varepsilon, a} \geq -C\varepsilon > -\delta/N.$$

This proves the claim. So for every $S \subset K^N$ we have

$$G_{\Omega_\varepsilon, S} \geq \sum_{a \in S} G_{\Omega_\varepsilon, a} > -\delta.$$

Fix $S \subset K^N$, we set $\hat{G} := G_{\Omega_\varepsilon, S}$ on $\Omega_\varepsilon \setminus \Omega$ while $\hat{G} := \max\{G_{\Omega_\varepsilon, S}, G_{\Omega, S} - \delta\}$ on Ω . Then $\hat{G} \in PSH^-(\Omega_\varepsilon)$ has logarithmic singularities at S . Therefore $\hat{G} \leq G_{\Omega_\varepsilon, S}$. The proof is then easily concluded. \square

Theorem 1.11 follows essentially from Theorem 1.5 and the following lemma which may be of independent interest.

Lemma 3.2. *Let Ω be a bounded hyperconvex domain in \mathbb{C}^n , $S \subset \Omega^N$, and K be a compact subset of $\Omega \setminus S$. Then for every $\varepsilon > 0$ and every relatively compact sub-domain Ω' such that $K \cup S \subset \Omega' \subset \Omega$ we can find $p \geq 1$, $\Omega' \subset \Omega^\varepsilon \Subset \Omega$ and a collection of holomorphic functions $f_1, \dots, f_{3n} \in \mathcal{I}_{\Omega, S}^p$ having the following properties:*

- (i) $\|f_j\|_{\overline{\Omega^\varepsilon}} < 1$ for every $1 \leq j \leq 3n$;
- (ii) *The following estimate holds on K*

$$G_{\Omega, S} - \varepsilon < \frac{1}{p} \max\{\log |f_1|, \dots, \log |f_{3n}|\} < G_{\Omega, S} + \varepsilon.$$

Proof. Choose Ω^ε such that $\Omega' \Subset \Omega^\varepsilon$ and that the following estimates hold on Ω^ε

$$G_{\Omega^\varepsilon, S} \leq G_{\Omega, S} + \varepsilon.$$

Next, we will prove that there exist an integer $p \geq 1$ and holomorphic functions $f_1, \dots, f_m \in \mathcal{I}_{\Omega, S}^p$ having the following properties:

- (i') $\|f_j\|_{\overline{\Omega^\varepsilon}} < 1$, for every $1 \leq j \leq m$.
- (ii') *The following estimates hold on K*

$$G_{\Omega, S} - \varepsilon/2 < \frac{1}{p} \max\{\log |f_1|, \dots, \log |f_m|\} < G_{\Omega, S} + \varepsilon/2.$$

To this end, following an idea of Demailly as in [15], we will use the Ohsawa-Takegoshi extension theorem in the following special form: *Let Ω be a bounded pseudoconvex domain, $\varphi \in PSH(\Omega)$ and $z \in \Omega$. Then for every complex number a , we can find a holomorphic function f in Ω such that $f(z) = a$ and*

$$\int_{\Omega} |f(w)|^2 e^{-\varphi(w)} dw \leq c_{\Omega, n} |a|^2 e^{-\varphi(z)},$$

where $c_{\Omega, n}$ depends only on the dimension n and the diameter of Ω .

Let $r > 0$ be the distance between $\partial\Omega$ and $\partial\Omega^\varepsilon$ and $A > 0$ be a constant which is smaller than the volume of the ball with radius r in \mathbb{C}^n . Choose an integer p so large such that

$$\varepsilon > -\frac{2}{p} \log\left(\frac{A}{c_{\Omega, n}}\right).$$

We apply the theorem to Ω , $z_0 \in K$, $\varphi = 2pG_{\Omega, S}$ and

$$a := \frac{\sqrt{A} e^{pG_{\Omega, S}(z_0)}}{\sqrt{c_{\Omega, n}}}.$$

Thus we can find a holomorphic function f on Ω such that

$$\int_{\Omega} |f(w)|^2 e^{-2pG_{\Omega,S}(w)} dw \leq A, f(z_0) = a.$$

The first inequality forces $f \in \mathcal{I}_{\Omega,S}^p$, the latter relation and the choice of p implies that

$$\frac{1}{p} \log |f(z_0)| > G_{\Omega,S}(z_0) - \varepsilon/2.$$

On the other hand, since $G_{\Omega,S} < 0$ on Ω we also get $\int_{\Omega} |f(w)|^2 dw < A$. By the sub-mean inequality applied to the subharmonic functions $|f|^2$ over balls of radius r with centers lying on $\partial\Omega^\varepsilon$, we conclude easily from this inequality that $\|f\|_{\overline{\Omega^\varepsilon}} < 1$. By a standard compactness argument we get a finite number of holomorphic functions $f_1, \dots, f_m \in \mathcal{I}_{\Omega,S}^p$ satisfying (i') and the left inequality in (ii'). For the other inequality, it suffices to note that by the choice of Ω^ε

$$\frac{1}{p} \max\{\log |f_1|, \dots, \log |f_m|\} \leq G_{\Omega^\varepsilon,S} \leq G_{\Omega,S} + \varepsilon/2.$$

We are done.

Now it is clear that the proof is complete in the case where $m \leq 3n$ by putting together (i') and (ii') (we can take trivially $f_{m+1} = \dots = f_{3n} = 0$ in this case). Suppose that $m \geq 3n+1$, following an idea in the proof of [1, Theorem 1], we proceed as follows. According to [8, Theorem 1], there exist polynomials g_1, \dots, g_n in \mathbb{C}^n such that

$$S = \{z \in \Omega : g_1(z) = \dots = g_n(z) = 0\}.$$

Choose polynomials g_{n+1}, \dots, g_m in \mathbb{C}^n such that any subset of n elements in the collection $\{g_1, \dots, g_m\}$ has S as their common zero set. This can be done by taking g_{n+1}, \dots, g_m to be sufficiently generic linear combinations of g_1, \dots, g_n . For $\eta_1, \dots, \eta_m \in \mathbb{C}$, we define

$$h_j := f_j + \eta_j g_j^p, 1 \leq j \leq m.$$

Obviously $h_j \in \mathcal{I}_{\Omega,S}^p$, the key step is to show that we can choose η_1, \dots, η_m so small such that the collection h_1, \dots, h_m has the the following additional properties:

- (a) $\|h_j\|_{\overline{\Omega^\varepsilon}} < 1$ for every $1 \leq j \leq m$;
- (b) $\{z \in \Omega \setminus S : |h_{j_1}(z)| = \dots = |h_{j_{3n+1}}(z)|\} = \emptyset$ for every $(3n+1)$ -tuple $(j_1, \dots, j_{3n+1}) \subset \{1, 2, \dots, m\}$;
- (c) $\|G_{\Omega,S} - w(z)\|_K < \varepsilon/4$, where $w(z) := \frac{1}{p} \max\{\log |h_1(z)|, \dots, \log |h_m(z)|\}$.

By the construction of h_1, \dots, h_m , the properties (a) and (c) are always verified if η_1, \dots, η_m are small enough. For the property (b), since the set of $(3n+1)$ -tuples $(j_1, \dots, j_{3n+1}) \subset \{1, 2, \dots, m\}$ is finite, for simplicity of notation, it suffices to treat the case where $j_1 = 1, \dots, j_{3n+1} = 3n+1$. For $1 \leq j \leq 3n+1$, denote by H_j the algebraic hypersurface $H_j := \{z \in \Omega : g_j(z) = 0\}$. Let

$$\Omega_j := (\Omega \cap H_1 \cap \dots \cap H_{j-1}) \setminus (H_j \cup \dots \cup H_{3n+1}),$$

$$\Delta_j := \{(w_j, \dots, w_{3n+1}) \in \mathbb{C}^{3n-j+2} : |w_j| = \dots = |w_{3n+1}|\}.$$

Then by the choice of g_j we have

$$\Omega \setminus S = \bigcup_{1 \leq j \leq n-1} \Omega_j.$$

It is also easy to check that

$$\dim_{\mathbb{R}}(\Omega_j) = 2(n - j + 1), \dim_{\mathbb{R}}(\Delta_j) = 3n - j + 3.$$

Consider the map $\Phi_j : \Omega_j \times \Delta_j \rightarrow \mathbb{C}^{3n-j+2}$ defined as

$$(z, w) \mapsto \left(\frac{w_j - f_j(z)}{g_j(z)^p}, \dots, \frac{w_{3n+1} - f_{3n+1}(z)}{g_{3n+1}(z)^p} \right).$$

Then Φ_j is a \mathcal{C}^∞ differentiable map from a real manifold of dimension $5n - 3j + 3$ into a real manifold of higher dimension $6n - 2j + 4$. This implies that $\Phi_j(\Omega_j \times \Delta_j)$ is a of Lebesgue measure 0 in \mathbb{C}^{3n-j+2} for every $j \in \{1, \dots, n\}$. Hence for every $\delta > 0$, there exists $\eta_1, \dots, \eta_{3n+1}$ such that $|\eta_j| < \delta$ for every $1 \leq j \leq 3n + 1$ and

$$(\eta_j, \dots, \eta_{3n+1}) \in \mathbb{C}^{3n-j+2} \setminus \Phi_j(\Omega_j \times \Delta_j), \quad \forall 1 \leq j \leq n - 1.$$

It implies our claim easily.

Now for every $1 \leq r \leq 3n$ and $s \geq 1$, we set

$$h_r^{(s)}(z) := \sum_{1 \leq j_1 < \dots < j_r \leq m} (h_{j_1}(z) \cdots h_{j_r}(z))^{s \frac{(3n)!}{r}}.$$

It follows from (a) that $\|h_r^{(s)}\|_\Omega < A$, where A depends only on n, m . Moreover, each $h_r^{(s)}$ belongs to $\mathcal{I}_{\Omega, S}^{p'_s}$, where $p'_s := ps(3n)!$. Consider the sequence of functions

$$w_s(z) := \frac{1}{p'_s} \max\{\log |h_1^{(s)}(z)|, \dots, \log |h_{3n}^{(s)}(z)|\}.$$

By the same reasoning as in [1, p. 1735], we will prove that there exists s_0 such that

$$\|G_{\Omega, S} - w_{s_0}\|_K < \varepsilon/2.$$

To this end, it suffices to approximate uniformly on K the function w by w_s for s large enough. We will do the lower bound for w_s , the upper bound is easier. Indeed, fix a point $z_0 \in K$. Then, by the above construction there exists $r = r(z_0) \leq 3n$ and a r tuple $J(z_0) := (j_1, \dots, j_r)$ such that $1 \leq j_1 < \dots < j_r \leq 3n$ and for any $i \notin J(z_0)$ we have

$$\lambda := |h_{j_1}(z_0)| = \dots = |h_{j_r}(z_0)| > |h_i(z_0)|.$$

Choose a small neighbourhood U_{z_0} of z_0 such that

$$d(z_0) := \max_I \sup_{\xi \in U_z} \left\{ \left| \frac{h_{i_1}(\xi) \cdots h_{i_r}(\xi)}{h_{j_1}(\xi) \cdots h_{j_r}(\xi)} \right| \right\} < 1,$$

where the maximum is taken over all r -tuples $I = (i_1, \dots, i_r)$ with $I \neq J$. By continuity we may assume that U_{z_0} satisfies the additional properties

$$(1 - \sigma)\lambda < |h_j(\xi)|, |w(z_0) - w(\xi)| < \sigma, \forall \xi \in U_{z_0}, \forall j \in J(z_0).$$

Here $\sigma \in (0, 1)$ will be chosen later on. Then for $\xi \in U_{z_0}$ we obtain the following estimates

$$\begin{aligned} |h_r^{(s)}(\xi)| &\geq |h_{j_1}(\xi) \cdots h_{j_r}(\xi)|^{\frac{s(3n)!}{r}} \left(1 - \sum_{I \neq J} \left| \frac{h_{i_1}(\xi) \cdots h_{i_r}(\xi)}{h_{j_1}(\xi) \cdots h_{j_r}(\xi)} \right|^s\right)^{\frac{(3n)!}{r}} \\ &\geq ((1 - \sigma)\lambda)^{s(3n)!} (1 - 2^m d(z)^s)^{\frac{(3n)!}{r}}. \end{aligned}$$

Then for $\xi \in U_{z_0}$ and $s \gg 1$ we get

$$w_s(\xi) \geq \frac{1}{ps(3n)!} \log |h_r^{(s)}(\xi)| \geq \frac{1}{p} (\log(1 - \sigma) + \log \lambda) + \frac{1}{rps} \log(1 - 2^m d(z_0)^s).$$

On the other hand, we also have

$$w(z_0) = \frac{1}{p} \log \lambda.$$

Thus, by shrinking U_{z_0} if necessary, we can find $\sigma \in (0, 1)$ and an integer $s(z_0) \geq 1$ such that

$$w_s(\xi) \geq w(\xi) - \varepsilon/4, \xi \in U_{z_0}, \forall s \geq s(z_0).$$

Now a standard compactness argument gives an integer $S(K)$ such that if $s \geq s(K)$ and $\xi \in K$ then

$$w_s(\xi) \geq w(\xi) - \varepsilon/4 \geq G_{\Omega, S}(\xi) - \varepsilon/2.$$

The proof is complete. \square

Proof of Theorem 1.11.

By Lemma 3.1, we can find $k_0 \geq 1$ and a hyperconvex domain Ω_ε containing $\overline{\Omega}$ such that the following inequality holds on $\Omega \setminus S_k$ for every $k \geq k_0$

$$G_{\Omega, S_k} - \varepsilon/2 < G_{\Omega_\varepsilon, S_k}.$$

By Theorem 1.5, there exists k_1 such that if $k \geq k_1$ then

$$\|G_{\Omega, S_k} - g\|_K < \varepsilon/2.$$

Using the above inequalities together with Lemma 3.2 (applied to $\Omega := \Omega_\varepsilon$, $S := S_k$ and $\Omega' := \Omega$, we can find p and holomorphic functions $f_{1,k}, \dots, f_{3n,k} \in \mathcal{I}_{\Omega_\varepsilon, S}^p$ satisfying the properties (i), (ii). Finally, by the same perturbation argument i.e., by adding small enough multiples of g_j^p , as was done in the proof of Lemma 3.2, we may achieve (iii).

We sketch the argument. For $1 \leq j \leq 3n$, denote by H_j the algebraic hypersurface $H_j := \{z \in \Omega_\varepsilon : g_j(z) = 0\}$. Let

$$\Omega_j := (\Omega_\varepsilon \cap H_1 \cap \cdots \cap H_{j-1}) \setminus (H_j \cup \cdots \cup H_{3n+1}).$$

Consider the map $\Psi_{j,k} : \Omega_j \mapsto \mathbb{C}^{3n-j+1}$ defined as

$$z \mapsto \left(-\frac{f_{j,k}(z)}{g_j(z)^p}, \dots, -\frac{f_{3n,k}(z)}{g_{3n}(z)^p} \right).$$

By counting the dimensions on both spaces we can get arbitrarily small $(\varepsilon_{1,k}, \dots, \varepsilon_{3n,k})$ such that $(\varepsilon_{j,k}, \dots, \varepsilon_{3n,k}) \notin \Psi_{j,k}(\Omega_j)$, for $1 \leq j \leq 3n$. Then it suffices to set $\tilde{f}_{j,k} := f_{j,k} + \varepsilon_j g_j^p$, $1 \leq j \leq 3n$. \square

Proof of Proposition 1.12.

Let $g \in PSH^-(\Omega)$ be an arbitrary limit point of G_{Ω, S_k} in the $L^1_{loc}(\Omega)$ topology. By definition of the multipole Green function and (ii) we obtain $G_{\Omega, S_k} \geq u_k$ for every k . It follows that $g \geq u$ a.e. on Ω . Since both functions are plurisubharmonic, we have $g \geq u$ everywhere on Ω . In particular $g \in \mathcal{F}(\Omega)$. By Theorem 1.5, $(dd^c g)^n$ is supported on $\pi(S)$ and

$$\int_{\Omega} (dd^c g)^n = N.$$

Now we apply Lemma 2.5 (a) to obtain

$$(dd^c u)^n(\{a\}) \geq (dd^c g)^n(\{a\}) \quad \forall a \in \pi(S).$$

This implies the following chain of inequalities

$$N \geq \int_{\Omega} (dd^c u)^n \geq \int_{\pi(S)} (dd^c u)^n \geq \int_{\pi(S)} (dd^c g)^n = N.$$

This forces

$$(dd^c u)^n = (dd^c g)^n = 0 \text{ on } \Omega \setminus \pi(S),$$

and $(dd^c u)^n(\{a\}) = (dd^c g)^n(\{a\}) \quad \forall a \in \pi(S)$. In other words, $(dd^c u)^n = (dd^c g)^n$ on Ω . Putting all this together and use Lemma 2.5 (ii) we have in fact $u = g$ on Ω . This implies that the whole sequence $\{G_{\Omega, S_k}\}$ converges to u in L^1_{loc} . Applying again Theorem 1.5, we conclude that the convergence occurs locally uniformly away from $\pi(S)$. \square

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